

FORM FACTOR REPRESENTATION OF THE CORRELATION FUNCTION OF THE TWO DIMENSIONAL ISING MODEL ON A CYLINDER

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Abstract

The correlation function of the two dimensional Ising model with the nearest neighbours interaction on the finite size lattice with the periodical boundary conditions is derived. The expressions similar to the form factor expansion are obtained both for the paramagnetic and ferromagnetic regions of coupling parameter. The peculiarities caused by finite size are analyzed. The scaling limit of the lattice form factor expansion is evaluated.

PACS number(s): 05.50.+q, 11.10.-z

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1 Introduction

Since the outstanding result of Montroll, Potts, Ward [1] there appeared many papers devoted to the problem of the spin-spin correlation function in the two dimensional Ising model (IM) with the nearest-neighbour interaction on the infinite size lattice (see e.g. refs. in [2]). The achievements in this field are mainly connected with the analysis of the scaling limit [3], [4], [5], because IM just in this limit is of interest from the quantum field theory point of view. The so-called form factor representation [6] for the two dimensional IM correlation function appears to be a crown of this activity:

$$\langle \sigma(\mathbf{r}_1) \sigma(\mathbf{r}_2) \rangle = \text{const} \sum_n g_n(r), \quad r = |\mathbf{r}_1 - \mathbf{r}_2|, \quad (1.1)$$

$$g_n(r) = \frac{1}{n!(2\pi)^n} \int_{-\infty}^{\infty} \prod_{i=1}^n \left(\frac{dq_i}{\omega_i} e^{-r\omega_i} \right) F_n^2[q], \quad (1.2)$$

$$F_n[q] = \prod_{i < j}^n \left(\frac{q_i - q_j}{\omega_i + \omega_j} \right), \quad \omega_i = \omega(q_i) = \sqrt{m^2 + q_i^2}. \quad (1.3)$$

In the ferromagnetic region of the coupling parameter the summation in (1.1) is extended over even n , in the paramagnetic — over odd. It is worth of noting that this representation was first evaluated [6] in the framework of the S -matrix approach [7] and then [5] by means of straightforward IM solution. The discovery of the form factor representation was very fruitful for the progress of exactly integrable quantum field theories [8].

The advantage of representation (1.1) consists in that the dynamical aspects of the system are separated from the kinematical ones: form factors squared are integrated over the phase volumes of the n -particle intermediate states. The form factor representation (1.1) clarifies the dynamics of the model but does not answer on the questions about the spectrum, sort and statistics of the particles that form the intermediate and asymptotic states of the system under consideration. The analysis of the model at finite (nonzero) temperature or in the finite volume is necessary for this purpose. One can observe an activity in this field in last years [9]–[15]. These works show in particular that the problem is complicated enough: the authors of refs. [14], [15], for example, call the conjectures of [11], [13] in question. I think that a simple exactly solvable lattice model example would be very useful for the business. Formally the finite temperature (in quantum field theory sense) means the finite size along the temporal axis and periodical boundary condition for boson fields or antiperiodical — for fermion fields. Meanwhile there is no representation which is analogous to the form factor one for the correlation function on the finite size lattice even for the Ising model.

This work, I hope, makes up a deficiency. I have calculated the IM spin-spin correlator on the lattice wrapped on a cylinder by the use of the classic methods of IM theory [16] adapted properly to the case of the finite sizes. The solution is expressed in the form similar to the form factor expansion (1.1)–(1.3). For reader's convenience I write the result just in the Introduction. If one considers it obvious he would be free of cumbersome mathematical

transformations. So, the expression obtained in this work is

$$\langle \sigma(\mathbf{r}_1) \sigma(\mathbf{r}_2) \rangle = \xi \xi_T e^{-r/\Lambda} \sum_n g_n(r), \quad (1.4)$$

$$g_n(r) = \frac{e^{-n/\Lambda}}{n!(N)^n} \sum_{[q]}^{(b)} \prod_{i=1}^n \left(\frac{e^{-r\gamma_i - \eta_i}}{\sinh \gamma_i} \right) F_n^2[q], \quad (1.5)$$

$$F_n[q] = \prod_{i < j}^n \frac{\sin((q_i - q_j)/2)}{\sinh((\gamma_i + \gamma_j)/2)}. \quad (1.6)$$

More precisely, the representation (1.4)–(1.6) corresponds to the correlation of spins posed on the line parallel to the axis of a cylinder with N sites on its circumference. The values ξ , ξ_T , Λ , γ_i , η_i , in eqs. (1.4)–(1.6) are defined further in Sections 2, 3, 4. Three of them — ξ_T , Λ and η_i — are specific cylinder circumference dependent values: $\ln \xi_T$, Λ^{-1} , η_i vanish if N tends to infinity. The appearance of the summation in (1.5) instead of the integration over phase volume is natural consequence of the size finiteness. It is widely believed that the underlying IM field is the fermion one: for instance the IM partition function is exactly the same as in the free Majorana fermion system. So, the boson spectrum of quasimomentum in Brillouin zone (this is denoted by the upper index in sum (1.5)) is surprising. Nevertheless it follows unambiguously from the calculations. It takes attention also that the lattice form factor (1.6) does not depends explicitly on the cylinder circumference. Note that similar expression for $F_n[q]$ at even n was found by authors of [5] for the infinite lattice case.

In Section 2 the model is formulated and the brief evaluation of the Toeplitz determinant representation for the IM correlation function is given. The corresponding Toeplitz matrix allowing for the finite size lattice is calculated. In Section 3 the lattice form factor representation for the correlation function is derived for the ferromagnetic domain of the coupling parameter and for the paramagnetic domain — in Section 4. The scaling limit is evaluated in Section 5. In Conclusion I discuss the relationship between the IM correlation function on a cylinder and quantum field Green function at finite temperature or in finite volume. I also comment on some prospect of generalization of the obtained results. In Appendix the factorised representations for the Toeplitz determinants of special form which are used for obtaining lattice form factor expansions are evaluated.

2 The Model

The Ising model with the nearest neighbour interaction on the square $M \times N$ lattice (see. Fig. 1) is defined by the hamiltonian $H[\sigma]$

$$-\beta H[\sigma] = K \sum_{\mathbf{r}} \sigma(\mathbf{r})(\nabla_x + \nabla_y) \sigma(\mathbf{r}), \quad K = \beta J,$$

where J is the coupling parameter, β is the inverse temperature; two dimensional vector $\mathbf{r} = (x, y)$ numerates the lattice sites: $x = 1, 2, \dots, M$, $y = 1, 2, \dots, N$; ∇_x , ∇_y are the one step shift operators

$$\nabla_x \sigma(x, y) = \sigma(x + 1, y), \quad \nabla_y \sigma(x, y) = \sigma(x, y + 1).$$

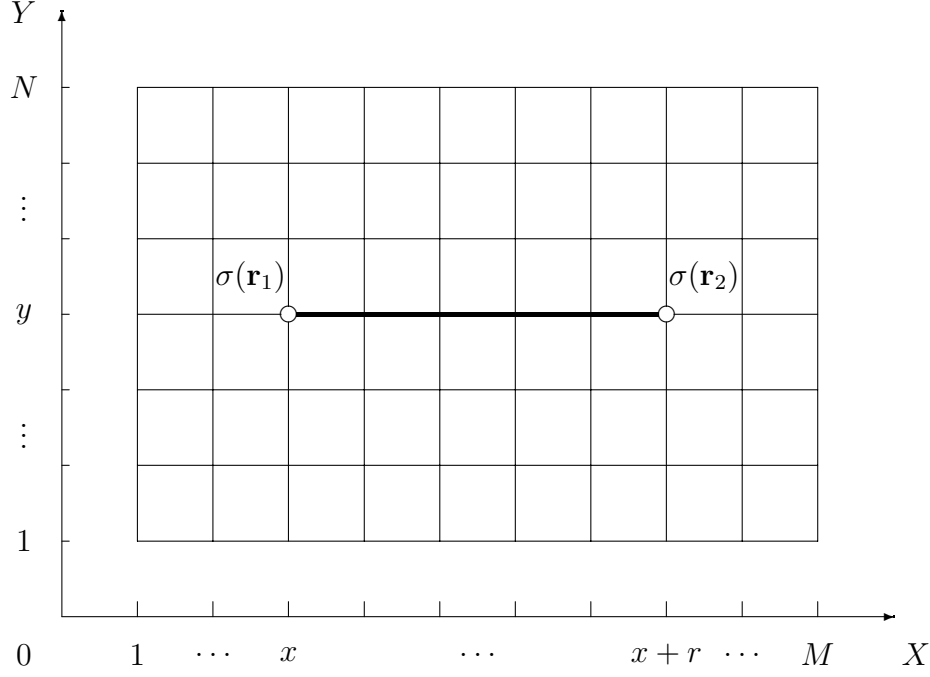


Fig. 1. The positions of the correlating Ising spins on the lattice.

The partition function of the model is

$$Z = \sum_{[\sigma]} e^{-\beta H[\sigma]}, \quad (2.1)$$

the pair correlation function is

$$\langle \sigma(\mathbf{r}_1) \sigma(\mathbf{r}_2) \rangle = Z^{-1} \sum_{[\sigma]} e^{-\beta H[\sigma]} \sigma(\mathbf{r}_1) \sigma(\mathbf{r}_2). \quad (2.2)$$

For the lattice with the periodical boundary conditions (wrapped on a torus) the partition function (2.1) can be expressed [17], [18] through the combination of the four Gauss Grassmann functional integrals with different boundary conditions

$$Z = (2 \cosh^2 K)^{MN} \frac{1}{2} (\mathcal{Q}^{(ff)} + \mathcal{Q}^{(bf)} + \mathcal{Q}^{(fb)} - \mathcal{Q}^{(bb)}), \quad (2.3)$$

where

$$\mathcal{Q} = \int d[\psi] e^{S[\psi]}, \quad d[\psi] = \prod_{\mathbf{r}} d\psi^1(\mathbf{r}) d\psi^2(\mathbf{r}) d\psi^3(\mathbf{r}) d\psi^4(\mathbf{r}).$$

The action $S[\psi]$ is the anticommuting quadratic form

$$S[\psi] = \frac{1}{2} \sum_{\mathbf{r}, \mathbf{r}'} \sum_{\nu, \rho=1}^4 \psi^\nu(\mathbf{r}) D^{\nu\rho}(\mathbf{r}, \mathbf{r}') \psi^\rho(\mathbf{r}') = \frac{1}{2} (\psi \hat{D} \psi), \quad (2.4)$$

$$\hat{D} = \begin{pmatrix} 0 & 1 + t\nabla_x & 1 & 1 \\ -1 - t\nabla_{-x} & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 + t\nabla_y \\ -1 & -1 & -1 - t\nabla_{-y} & 0 \end{pmatrix}, \quad t = \tanh K. \quad (2.5)$$

The indices (f, b) in eq. (2.3) shows the type of the boundary conditions for the shift operators in the matrix (2.5).

- “ f ” – antiperiodical: $(\nabla_x^{(f)})^M = -1$, $(\nabla_y^{(f)})^N = -1$;
the corresponding quasimomentum runs over halfinteger values in Brillouin zone
 $(-\pi, \pi)$ (in $2\pi/M$ unites for p_x and $2\pi/N$ – for p_y)
- “ b ” – periodical: $(\nabla_x^{(b)})^M = 1$, $(\nabla_y^{(b)})^N = 1$;
the corresponding components of quasimomentum runs over integer values.

The correlation function (2.2) can be also presented in terms of the Grassmann functional integrals with the Gauss distribution [18]

$$\langle \sigma(\mathbf{r}_1) \sigma(\mathbf{r}_2) \rangle = t^r \frac{\mathcal{Q}_d^{(ff)} + \mathcal{Q}_d^{(bf)} + \mathcal{Q}_d^{(fb)} - \mathcal{Q}_d^{(bb)}}{\mathcal{Q}^{(ff)} + \mathcal{Q}^{(bf)} + \mathcal{Q}^{(fb)} - \mathcal{Q}^{(bb)}}, \quad (2.6)$$

where \mathcal{Q}_d is the functional integral

$$\mathcal{Q}_d = \int d[\psi] e^{S_d[\psi]}. \quad (2.7)$$

The action with the defect denoted by $S_d[\psi]$ in (2.7) differs from (2.4) in that the parameter t is replaced by t^{-1} on the links along the path which connects the sites \mathbf{r}_1 and \mathbf{r}_2 .

I consider the special case when the sites \mathbf{r}_1 and \mathbf{r}_2 are situated on the line parallel to the horizontal axis as is shown on Fig. 1:

$$\mathbf{r}_1 = (x, y), \quad \mathbf{r}_2 = (x + r, y), \quad |\mathbf{r}_1 - \mathbf{r}_2| = r.$$

The action with the defect in this case has the form

$$S_d[\psi] = S[\psi] + \Delta S[\psi],$$

where

$$\Delta S[\psi] = (t^{-1} - t) \sum_{x'=1}^r \psi^1(x + x' - 1, y) \psi^2(x + x', y).$$

The ratio in the r.h.s. of (2.6) simplifies

$$\langle \sigma(\mathbf{r}_1) \sigma(\mathbf{r}_2) \rangle = t^r \frac{\mathcal{Q}_d^{(ff)}}{\mathcal{Q}^{(ff)}} [1 + o(e^{-M/N})], \quad (2.8)$$

if at least one of the lattice sizes is much larger than other, i.e. torus degenerates to cylinder: in our case $M/N \gg 1$. Note, that only the terms with the antiperiodical boundary conditions survive.

After some manipulations with the functional integrals the ratio in the r.h.s. of (2.8) can be transformed to the functional integral over Grassmann field which is defined just on the line connecting the sites \mathbf{r}_1 and \mathbf{r}_2

$$\langle \sigma(\mathbf{r}_1) \sigma(\mathbf{r}_2) \rangle = \int d[\chi] e^{(\chi^1 A \chi^2)} = \det A, \quad (2.9)$$

$$\begin{aligned} d[\chi] &= \prod_{x=0}^{r-1} d\chi_x^1 d\chi_x^2, \quad (\chi^1 A \chi^2) = \sum_{x,x'} \chi_x^1 A_{x,x'} \chi_{x'}^2, \\ A_{x,x'} &= \frac{1}{MN} \sum_{\mathbf{p}}^{(f)} \frac{e^{ip_x(x-x')} [2t(1+t^2) - (1-t^2)(e^{ip_x} + t^2 e^{-ip_x})]}{(1+t^2)^2 - 2t(1-t^2)(\cos p_x + \cos p_y)}, \\ x, x' &= 0, 1, \dots, r-1. \end{aligned} \quad (2.10)$$

I recall that the index (f) points the antiperiodicity: the momentum's components run over halfinteger values – fermion spectrum

$$\sum_{p_x}^{(f)} v(p_x) = \sum_{l=1}^M v\left(\frac{2\pi}{M}(l + \frac{1}{2})\right), \quad \sum_{p_y}^{(f)} v(p_y) = \sum_{l=1}^N v\left(\frac{2\pi}{N}(l + \frac{1}{2})\right).$$

Now let us rewrite the expression (2.10) in appropriate for the Toeplitz determinant theory manner. From the tabulated formulas [20]

$$\prod_{p_y}^{(f)} 2(\cosh \gamma - \cos p_y) = e^{N\gamma}(1 + e^{-N\gamma})^2, \quad \prod_{p_y}^{(b)} 2(\cosh \gamma - \cos p_y) = e^{N\gamma}(1 - e^{-N\gamma})^2, \quad (2.11)$$

it follows in particular

$$\frac{1}{N} \sum_{p_y}^{(f)} \frac{1}{\cosh \gamma - \cos p_y} = \frac{\tanh(N\gamma/2)}{\sinh \gamma}, \quad \frac{1}{N} \sum_{p_y}^{(b)} \frac{1}{\cosh \gamma - \cos p_y} = \frac{\coth(N\gamma/2)}{\sinh \gamma}. \quad (2.12)$$

At the cylinder case $M \rightarrow \infty$, $N = \text{const}$ the summation over p_x turns into the integral

$$\frac{1}{M} \sum_{p_x}^{(f)} v(e^{-ip_x}) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} v(z) + o(e^{-M\mu}). \quad (2.13)$$

Summing up (2.10) over p_y by use of (2.12) and accounting for (2.13) one can obtain for the matrix $A_{x,x'}$

$$A_{x,x'} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} z^{-x+x'} A(z), \quad (2.14)$$

where the kernel $A(z)$ is

$$A(z) = \sqrt{\frac{(1 - e^{-\gamma\pi} z)(1 - e^{-\gamma_0} z^{-1})}{(1 - e^{-\gamma_0} z)(1 - e^{-\gamma\pi} z^{-1})}} \cdot T(z), \quad z = e^{-ip}, \quad (2.15)$$

and

$$T(z) = \tanh[(N\gamma(p))/2], \quad (2.16)$$

$$\gamma_0 = |2K + \ln t|, \quad \gamma_\pi = 2K - \ln t. \quad (2.17)$$

The function $\gamma(p)$ in (2.16) is defined by the following equation

$$4 \sinh^2 \frac{\gamma(p)}{2} = \mu^2 + 4 \sin^2 \frac{p}{2}, \quad (2.18)$$

where

$$\mu^2 = 2(\sqrt{\sinh 2K} - 1/\sqrt{\sinh 2K})^2. \quad (2.19)$$

It follows from eqs. (2.18), (2.19) in particular

$$\gamma(0) = \gamma_0, \quad \gamma(\pi) = \gamma_\pi, \quad \cosh \gamma_\pi - \cosh \gamma_0 = 2.$$

The function $T(z)$ (2.16) tends to unity at the thermodynamic limit $N \rightarrow \infty$ (more precise $N\mu \gg 1$), and the kernel $A(z)$ (2.15) turns into the classic IM theory expression. In this limit the value $K = K_c$ is the critical point: the specific heat and correlation length diverge. At the finite N the corresponding singularities are smoothed and there is no phase transition. Nevertheless also in this case the value $K = K_c$ is notable. The calculation technique based on Wiener–Hopf sum equation “distinguishes” the ferro- and paramagnetic regions and the expression for the correlation function at $K > K_c$ differs from that at $K < K_c$. Just in this sense I shall mean further the words “critical point”, “phase” and so on. Note that the following equalities take place at the critical point

$$\sinh 2K_c = 1, \quad \mu = 0, \quad \gamma_0 = 0, \quad \gamma_\pi = 2 \ln(\sqrt{2} + 1).$$

3 The Ferromagnetic region: $K > K_c$

The dependence of the correlation function (2.9)

$$\langle \sigma(\mathbf{r}_1) \sigma(\mathbf{r}_2) \rangle = \det A^{(r)} \quad (3.1)$$

on the distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$ enters through the matrix dimension denoted by upper index (r) in the r.h.s. of (3.1). The dependence on the cylinder circumference N comes into the function $T(z)$ (2.16) and the dependence on the coupling parameter K is defined by the parameter μ (2.19) in the eq. (2.18) for the function $\gamma(p)$.

The kernel (2.14) of the matrix (2.9) $A^{(r)}$ has to be represented in the factorized form according to the method of the determinant calculation which we use (see (A6), (A7) in the Appendix),

$$A(z) = P(z)Q(z^{-1}),$$

where the functions $P(z)$ and $Q(z)$ are analytic ones inside the $|z| = 1$ circle. It can be formally done by use of the projection operators (A8) that are defined in the Appendix

$$\begin{aligned} 2 \ln P(z) &= \ln(1 - e^{-\gamma_\pi z}) - \ln(1 - e^{-\gamma_0 z}) + 2\mathcal{P} \ln T(z), \\ 2 \ln Q(z^{-1}) &= \ln(1 - e^{-\gamma_0 z^{-1}}) - \ln(1 - e^{-\gamma_\pi z^{-1}}) + 2\tilde{\mathcal{P}} \ln T(z). \end{aligned}$$

Otherwise accounting for the explicit form (2.16) one can express $T(z)$ through the ratio of the products with help of eqs. (2.11)

$$T^2(z) = \frac{\prod_q^{(b)} [\cosh \gamma(p) - \cos q]}{\prod_q^{(f)} [\cosh \gamma(p) - \cos q]}. \quad (3.2)$$

It follows from (2.18)

$$\cosh \gamma(p) - \cos q = 2 + \frac{\mu^2}{2} - \cos q - \cos p = \frac{1}{2} e^{\gamma(q)} (1 - e^{-\gamma(q)} z) (1 - e^{-\gamma(q)} z^{-1}),$$

so eq. (3.2) turns into the form

$$T^2(z) = \frac{\prod_q^{(b)} e^{\gamma(q)} (1 - e^{-\gamma(q)} z) (1 - e^{-\gamma(q)} z^{-1})}{\prod_q^{(f)} e^{\gamma(q)} (1 - e^{-\gamma(q)} z) (1 - e^{-\gamma(q)} z^{-1})}. \quad (3.3)$$

Owing to the momentum q runs the entire Brillouin zone all the factors of the r.h.s. (3.3) are double except for the values $q = 0$ and $q = \pi$ in the nominator. So the functions $A(z)$, $P(z)$ and $Q(z^{-1})$ have no branch points rather the simple poles and zeroes:

$$P(z) = \exp\left(\frac{\gamma_0 - \gamma_\pi}{2}\right) \frac{\prod_{0 < q \leq \pi}^{(b)} (e^{\gamma(q)} - z)}{\prod_{0 < q < \pi}^{(f)} (e^{\gamma(q)} - z)}, \quad Q(z^{-1}) = \frac{\prod_{0 \leq q < \pi}^{(b)} (1 - e^{-\gamma(q)} z^{-1})}{\prod_{0 < q < \pi}^{(f)} (1 - e^{-\gamma(q)} z^{-1})}. \quad (3.4)$$

According to (A25), (A28)

$$\det A^{(r)} = e^{h(r)} \sum_{l=0}^{\infty} g_{2l}(r). \quad (3.5)$$

Let us write the function $h(r)$ as the sum of three terms

$$h(r) = -r/\Lambda + \ln \xi + \ln \xi_T. \quad (3.6)$$

It follows from the eqs. (A23) and (2.15)

$$\Lambda^{-1} = -[\ln P(0) + \ln Q(0)] = -\frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} \ln T(z) = \frac{1}{\pi} \int_0^\pi dp \ln \coth(N\gamma(p)/2). \quad (3.7)$$

The parameter Λ has the sense of the coherence length. Its asymptotic behaviour at large N is

$$\Lambda \simeq e^{N\gamma_0} \sqrt{\frac{\pi N}{2 \sinh \gamma_0}}. \quad (3.8)$$

One can see from (3.8) that the coherence length grows very rapidly when the cylinder circumference increases and $\gamma_0 \neq 0$. The lattice can be considered as infinite at $N\gamma_0 \gg \max[\ln(N\gamma_0), \ln(r\gamma_0)]$. It follows also from (A23) that

$$\ln(\xi\xi_T) = \frac{1}{2\pi i} \oint_{|z|=1} dz \ln Q(z^{-1}) \frac{\partial}{\partial z} \ln P(z) = \frac{1}{(2\pi i)^2} \oint_{|z_2| > |z_1|} d(z_1 z_2) \frac{\ln A(z_1) \ln A(z_2)}{(z_2 - z_1)^2}, \quad (3.9)$$

or separately for the $\ln \xi$ and $\ln \xi_T$

$$\ln \xi = \frac{1}{4} \ln \left[\frac{\sinh \gamma_0 \cdot \sinh \gamma_\pi}{\sinh^2((\gamma_0 + \gamma_\pi)/2)} \right], \quad (3.10)$$

$$\ln \xi_T = \frac{1}{(2\pi i)^2} \oint_{|z_2| > |z_1|} \frac{d(z_1 z_2)}{(z_2 - z_1)^2} \ln T(z_1) \ln T(z_2). \quad (3.11)$$

The r.h.s. of the eq. (3.10) does not depend on N and is singular at the critical point

$$\ln \xi \simeq \frac{1}{4} \ln \mu \quad \text{at} \quad \mu \rightarrow 0. \quad (3.12)$$

For the calculation $\ln \xi_T$ let us return to the momentum variables and integrate (3.11) by parts. The result is

$$\ln \xi_T = \frac{N^2}{2\pi^2} \int_0^\pi \frac{dp dq \gamma'(p) \gamma'(q)}{\sinh(N\gamma(p)) \sinh(N\gamma(q))} \ln \left| \frac{\sin((p+q)/2)}{\sin((p-q)/2)} \right|$$

or

$$\ln \xi_T = \frac{N^2}{2\pi^2} \int_{\gamma_0}^{\gamma_\pi} \frac{d\gamma_p d\gamma_q}{\sinh(N\gamma(p)) \sinh(N\gamma(q))} \ln \left| \frac{\sin((p+q)/2)}{\sin((p-q)/2)} \right|. \quad (3.13)$$

Differentiating eq. (3.13) with respect to μ one can obtain

$$\frac{\partial}{\partial \mu^2} \ln \xi_T = -\frac{1}{2} [c_1^2(\mu) - c_2^2(\mu)], \quad (3.14)$$

where

$$\begin{aligned} c_1(\mu) &= \frac{N}{2\pi} \int_{\gamma_0}^{\gamma_\pi} \frac{d\gamma}{\sinh(N\gamma)} \sqrt{\frac{\cosh \gamma_\pi - \cosh \gamma}{\cosh \gamma - \cosh \gamma_0}}, \\ c_2(\mu) &= \frac{N}{2\pi} \int_{\gamma_0}^{\gamma_\pi} \frac{d\gamma}{\sinh(N\gamma)} \sqrt{\frac{\cosh \gamma - \cosh \gamma_0}{\cosh \gamma_\pi - \cosh \gamma}}. \end{aligned} \quad (3.15)$$

Deriving eqs. (3.14), (3.15) one has to account for that the integrand of (3.13) vanishes on the boundaries of the integration region. Note also that the derivatives of p and q are

$$\frac{\partial p}{\partial \mu^2} = -\frac{1}{2 \sin p}, \quad \frac{\partial q}{\partial \mu^2} = -\frac{1}{2 \sin q},$$

that follows from (2.18). For $N\gamma_0 \gg 1$ the asymptotics of $c_1(\mu)$ and $c_2(\mu)$ are

$$c_1(\mu) \simeq \frac{1}{\cosh(N\gamma_0)} \sqrt{\frac{N}{2\pi \sinh \gamma_0}}, \quad c_2(\mu) \simeq \frac{1}{4 \cosh(N\gamma_0)} \sqrt{\frac{\sinh \gamma_0}{2\pi N}}. \quad (3.16)$$

Collecting (3.14)–(3.16) one can obtain

$$\ln \xi_T = \int_{\mu}^{\infty} d\nu \nu [c_1^2(\nu) - c_2^2(\nu)] \simeq \frac{1}{\pi} e^{-2N\gamma_0}.$$

So, we see that $\xi_T \rightarrow 1$ if $N \rightarrow \infty$.

The integral (3.13) diverges at the critical point. This means that $\ln \xi_T$ as a function of μ has a singularity. It is easy to obtain at $\mu \rightarrow 0$

$$c_1(\mu) \simeq 1/2\mu, \quad \frac{\partial \ln \xi_T}{\partial \mu^2} = -\frac{1}{8\mu^2}$$

and consequently

$$\ln \xi_T = -\frac{1}{4} \ln \mu + \text{const.} \quad (3.17)$$

Therefore, one can see from (3.12), (3.17) that the function $\ln(\xi \xi_T)$ has no singularity at $\mu = 0$ if N is finite.

Let now evaluate the expansion in the r.h.s. of (3.5)

$$G_F(r) = \sum_{l=0}^{\infty} g_{2l}(r). \quad (3.18)$$

The coefficients $g_{2l}(r)$ are expressed through the $2l$ -multiple contour integrals (A30) as it is shown in the Appendix

$$g_{2l}(r) = \frac{(-1)^l}{l!l!(2\pi i)^{2l}} \oint_{|z_i|<1} \frac{\prod_{i=1}^{2l} (dz_i z_i^r) \prod_{i=1}^{l-1} \prod_{j=i+1}^l [(z_{2i-1} - z_{2j-1})^2 (z_{2i} - z_{2j})^2]}{\prod_{i=1}^l \prod_{j=1}^l (1 - z_{2i-1} z_{2j})^2} \frac{\prod_{i=1}^l W(z_{2i-1})}{\prod_{i=1}^l W(z_{2i}^{-1})}. \quad (3.19)$$

The integral (3.19) is defined by the concrete form of the function $W(z) = P(z)/Q(z^{-1})$: in our case this is the eq. (3.4). The integrand in the r.h.s. of (3.19) is analytic at $|z_i| < 1$ except for the singularities which the functions $W(z_i)$ and $W^{-1}(z_i^{-1})$ possess. To separate explicitly the singularities inside the circle $|z| = 1$ these functions ought to be rewritten in the following way

$$W(z) = \frac{P(z)}{Q(z^{-1})} = \sqrt{\frac{(1 - e^{-\gamma\pi} z)(1 - e^{-\gamma\pi} z^{-1})}{(1 - e^{-\gamma_0} z)(1 - e^{-\gamma_0} z^{-1})}} \frac{P_T^2(z)}{T(z)}, \quad (3.20)$$

$$W^{-1}(z^{-1}) = \frac{Q(z)}{P(z^{-1})} = \sqrt{\frac{(1 - e^{-\gamma_0} z)(1 - e^{-\gamma_0} z^{-1})}{(1 - e^{-\gamma\pi} z)(1 - e^{-\gamma\pi} z^{-1})}} \frac{Q_T^2(z)}{T(z)}. \quad (3.21)$$

The analytic at $|z| < 1$ functions $P_T(z)$ and $Q_T(z)$ are defined by the equations

$$\ln P_T(z) = \mathcal{P} \ln T(z) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz'}{z' - z} \ln \tanh(N\gamma(p')/2), \quad (3.22)$$

$$\ln Q_T(z^{-1}) = \tilde{\mathcal{P}} \ln T(z) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz'}{z - z'} \ln \tanh(N\gamma(p')/2), \quad (3.23)$$

$$z' = e^{-ip'}.$$

Note that it follows from (3.7), (3.22), (3.23)

$$\ln P_T(0) = -1/\Lambda, \quad \ln Q_T(0) = 0, \quad P_T(z) = P_T(0) \cdot Q_T(z).$$

With allowance of these eqs. one can obtain instead of (3.20), (3.21)

$$W(z) = e^{(\gamma_0 - \gamma_\pi)/2 - 2/\Lambda} (\cosh \gamma_\pi - \cos p) Q_T^2(z) \frac{\coth(N\gamma(p)/2)}{\sinh \gamma(p)}, \quad (3.24)$$

$$W^{-1}(z^{-1}) = e^{(\gamma_\pi - \gamma_0)/2} (\cosh \gamma_0 - \cos p) Q_T^2(z) \frac{\coth(N\gamma(p)/2)}{\sinh \gamma(p)}, \quad (3.25)$$

$$\frac{\coth(N\gamma(p)/2)}{\sinh \gamma(p)} = \frac{1}{N} \sum_q^{(b)} \frac{1}{\cosh \gamma(q) - \cos p}. \quad (3.26)$$

It is seen from (3.24)–(3.26) that the singularities inside the integration contours of (3.19) are the simple poles at the points

$$z_{(k)} = e^{-\gamma(q_{(k)})}, \quad q_{(k)} = \frac{2\pi k}{N}, \quad k = 1, \dots, N,$$

corresponding to the boson spectrum of the quasimomentum q . Accounting for

$$\frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} \frac{v(z)}{\cosh \gamma(q) - \cos p} = \frac{v(e^{-\gamma(q)})}{\sinh \gamma(q)},$$

where $v(z)$ is analytic at $|z| < 1$, the integral (3.19) is fulfilled by residua. The result is

$$g_{2l}(r) = \frac{e^{-2l/\Lambda}}{(2l)! N^{2l}} \sum_{[q]}^{(b)} \prod_{i=1}^{2l} \left(\frac{e^{-r\gamma_i - \eta_i}}{\sinh \gamma_i} \right) \mathcal{G}_{2l}[q], \quad (3.27)$$

where

$$\begin{aligned} \mathcal{G}_{2l}[q] = & C_l^{2l} \prod_{i=1}^{l-1} \prod_{j=i+1}^l [(e^{-\gamma_{2i-1}} - e^{-\gamma_{2j-1}})^2 (e^{-\gamma_{2i}} - e^{-\gamma_{2j}})^2] \times \\ & \times \frac{\prod_{i=1}^l [(1 + \cos q_{2i-1})(1 - \cos q_{2i})]}{\prod_{i=1}^{2l} e^{\gamma_i} \prod_{i=1}^l \prod_{j=1}^l (1 - e^{-\gamma_{2i} - \gamma_{2j}})^2}, \end{aligned} \quad (3.28)$$

$$\gamma_i = \gamma(q_i), \quad \eta_i = \eta(q_i) = -\ln Q_T^2(e^{-\gamma(q_i)}), \quad (3.29)$$

$$\sum_{[q]}^{(b)} = \sum_{q_1}^{(b)} \sum_{q_2}^{(b)} \cdots \sum_{q_{2l}}^{(b)},$$

$$C_l^{2l} = \frac{(2l)!}{l!l!} \quad - \quad \text{is the binomial coefficient.}$$

Using the equalities

$$\frac{1}{2}(e^{-\gamma_i} - e^{-\gamma_j})(1 - e^{\gamma_i + \gamma_j}) = \cosh \gamma_i - \cosh \gamma_j = \cos q_j - \cos q_i,$$

the eq. (3.28) can be transformed to

$$\mathcal{G}_{2l}[q] = \frac{V_{2l}[q]}{\prod_{i < j}^{2l} \sinh^2((\gamma_i + \gamma_j)/2)},$$

where

$$\begin{aligned} V_{2l}[q] &= 2^{-2l^2} C_l^{2l} \prod_{i=1}^{l-1} \prod_{j=i+1}^l [(\cos q_{2i-1} - \cos q_{2j-1})^2 (\cos q_{2i} - \cos q_{2j})^2] \times \\ &\times \prod_{i=1}^l (1 + \cos q_{2i-1})(1 - \cos q_{2i}). \end{aligned} \quad (3.30)$$

The expression (3.30) is not symmetric with respect to changes the summation variables q_i with even subscripts by that with odd subscripts. The summation extracts the symmetric part of the function $V_{2l}[q]$ which is denoted by $V_{2l}^{\text{sim}}[q]$

$$V_{2l}^{\text{sim}}[q] = \frac{1}{C_l^{2l}} \sum_{\text{P}} V_{2l}[q],$$

where the sum over all permutations of q_{2i-1} and q_{2j} (their number is just C_l^{2l}) is denoted by \sum_{P} . The equality

$$V_{2l}^{\text{sim}}[q] = U_{2l}^{\text{even}}[q], \quad (3.31)$$

is the crucial for the final result. Here the even (with respect to $q_i \leftrightarrow -q_i$) part of the product

$$U_{2l}[q] = \prod_{i < j}^{2l} \sin^2 \left(\frac{q_i - q_j}{2} \right),$$

is denoted by $U_{2l}^{\text{even}}[q]$

$$U_{2l}^{\text{even}}[q] = 2^{-2l} \sum_{[\sigma=\pm 1]} \prod_{i < j}^{2l} \sin^2 \left(\frac{\sigma_i q_i - \sigma_j q_j}{2} \right).$$

Consequently the function $V_{2l}[q]$ under summation (3.27) can be replaced by the function $U_{2l}[q]$. This makes it possible to present the coefficient $g_{2l}(r)$ in the form similar to that of eq. (1.2)

$$n = 2l,$$

$$g_n(r) = \frac{e^{-n/\Lambda}}{n!N^n} \sum_{[q]}^{(b)} \prod_{i=1}^n \left(\frac{e^{-r\gamma_i - \eta_i}}{\sinh \gamma_i} \right) F_n^2[q_i], \quad (3.32)$$

where $F_n[q_i]$ is the lattice analog of the form factor (1.3)

$$F_n[q_i] = \prod_{i < j}^n \left[\frac{\sin((q_i - q_j)/2)}{\sinh((\gamma_i + \gamma_j)/2)} \right], \quad F_0[q] = 1. \quad (3.33)$$

I emphasize once more that the momentum spectrum over which the summation (3.32) is extended (the intermediate states summing up) occurs to be boson one contrary to the fermion spectrum by which the initial values were defined, in particular the matrix $A_{x,x'}$ (2.10).

The function $\eta_i = \eta(q_i)$ that is contained in (3.32), decreases rapidly when the cylinder circumference grows. It follows from its definition (3.29)

$$\eta(q) = -\frac{1}{\pi i} \oint_{|z|=1} \frac{dz \ln T(z)}{e^{\gamma(q)} - z} = \frac{1}{\pi} \int_0^\pi \frac{dp(1 - e^{-\gamma(q)} \cos p)}{\cosh \gamma(q) - \cos p} \ln \coth(N\gamma(p)/2). \quad (3.34)$$

One can find at $N\gamma_0 \gg 1$

$$\eta(q) \simeq \frac{4}{e^{\gamma(q)} - 1} \frac{e^{-N\gamma_0}}{\sqrt{2\pi N \sinh \gamma_0}},$$

and $\eta(q) \rightarrow 0$ if $N \rightarrow \infty$.

4 The Paramagnetic region: $K < K_c$

The correlation function is defined uniform through the determinant $\det A^{(r)}$ in the whole region of parameters (including $K < K_c$). But the method of its calculation is to be modified because the Wiener-Hopf sum equation technique, which is used, demands the Toeplitz matrix kernel of the appropriate form. Both the kernel and the logarithm of the kernel have to satisfy the Loran expansion condition. Meanwhile, the matrix kernel (2.10) is reforming compare to the ferromagnetic case (2.15), when K goes across K_c

$$A(z) = z^{-1}B(z),$$

where

$$B(z) = \sqrt{\frac{(1 - e^{-\gamma_0}z)(1 - e^{-\gamma_\pi}z)}{(1 - e^{-\gamma_0}z^{-1})(1 - e^{-\gamma_\pi}z^{-1})}} T(z), \quad (4.1)$$

so, rather the function $B(z)$ than $A(z)$ possesses appropriate analytical properties. The matrix $A^{(r)}$ now has the following form

$$A_{x,x'} = -\frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} z^{-x+x'-1} B(z). \quad (4.2)$$

Factorizing the kernel (4.1)

$$B(z) = P(z) \cdot Q(z^{-1}),$$

where the functions $P(z)$ and $Q(z)$ are analytic inside the circle $|z| < e^{\gamma_0}$ one can obtain

$$P(z) = \exp\left(-\frac{\gamma_0 + \gamma_\pi}{2}\right) \frac{\prod_{0 \leq q \leq \pi}^{(b)} (e^{\gamma(q)} - z)}{\prod_{0 < q < \pi}^{(f)} (e^{\gamma(q)} - z)}, \quad Q(z^{-1}) = \frac{\prod_{0 < q < \pi}^{(b)} (1 - e^{-\gamma(q)} z^{-1})}{\prod_{0 < q < \pi}^{(f)} (1 - e^{-\gamma(q)} z^{-1})}. \quad (4.3)$$

Comparing with the factorized representation in the ferromagnetic case one can see that the corresponding products (3.4) and (4.3) differ from each other by one term: the factor $(e^{\gamma_0} - z)$ is appeared in $P(z)$ and the factor $(1 - e^{-\gamma_0} z^{-1})$ is disappeared in $Q(z^{-1})$. It follows from (4.3)

$$P(0) = P_T(0) = e^{-1/\Lambda}, \quad Q(0) = Q_T(0) = 1,$$

where the functions $P_T(z)$ and $Q_T(z)$ are the same as in the ferromagnetic case (3.22), (3.23).

The matrix (4.2) has the structure (A31) considered in the Appendix. Therefore its determinant can be represented according to the eqs. (A32)–(A35) by the following way

$$\det A^{(r)} = e^{h(r+1)} F(r), \quad (4.4)$$

$$F(r) = \sum_{l=0}^{\infty} f_{2l+1}(r).$$

The function $h(r+1)$ is given by eq. (A23). Writing it similarly to (3.6) and allowing for the difference between the definitions (4.3) and (3.4) for the corresponding functions $P(z)$ and $Q(z)$ one can obtain

$$h(r+1) = -(r+1)/\Lambda + \ln(\xi \xi_T) + \ln(1 - e^{-\gamma_0 - \gamma_\pi}), \quad (4.5)$$

where the values Λ , ξ and ξ_T are the same as in the ferromagnetic phase (3.7), (3.10) and (3.11). With the allowance (4.5) the eq. (4.4) can be rewritten as follows

$$\det A^{(r)} = \sinh((\gamma_0 + \gamma_\pi)/2) e^{h(r)} G_P(r), \quad (4.6)$$

where

$$G_P(r) = 2e^{-1/\Lambda - (\gamma_0 + \gamma_\pi)/2} F(r) = \sum_{l=0}^{\infty} g_{2l+1}(r), \quad (4.7)$$

and for the coefficients $g_{2l+1}(r)$ in the expansion (4.7) one obtains from (A35)

$$g_{2l+1}(r) = 2e^{-1/\Lambda - (\gamma_0 + \gamma_\pi)/2} \frac{(-1)^l}{l!(l+1)!} (2\pi i)^{2l+1} \oint_{|z_i| < 1} \prod_{i=1}^{2l+1} (dz_i z_i^r) \times \\ \times \frac{\prod_{i=1}^{l-1} \prod_{j=i+1}^l (z_{2i} - z_{2j})^2 \prod_{i=1}^l \prod_{j=i+1}^{l+1} (z_{2i-1} - z_{2j-1})^2 \prod_{i=1}^l (z_{2i} W(z_{2i}))}{\prod_{i=1}^l \prod_{j=0}^l (1 - z_{2i} z_{2j+1})^2 \prod_{i=0}^l (z_{2i+1} W(z_{2i+1}^{-1}))}. \quad (4.8)$$

The functions $W(z) = P(z)/Q(z^{-1})$ and $W^{-1}(z^{-1}) = Q(z)/P(z^{-1})$ entering the integrals (4.8) have the following form in this case

$$W(z) = 2e^{-2/\Lambda - (\gamma_0 + \gamma_\pi)/2} Q_T^2(z) \sinh^2 \gamma(p) \frac{1}{N} \sum_q^{(b)} \frac{1}{\cosh \gamma(q) - \cos p}, \quad (4.9)$$

$$W^{-1}(z^{-1}) = \frac{1}{2} e^{(\gamma_0 + \gamma_\pi)/2} Q_T^2(z) \frac{1}{N} \sum_q^{(b)} \frac{1}{\cosh \gamma(q) - \cos p}. \quad (4.10)$$

It follows from (4.9), (4.10) that the integrand in (4.8) has only the simple poles inside the integration contour at the points

$$z_{(k)} = e^{-\gamma(q_{(k)})}, \quad q_{(k)} = \frac{2\pi k}{N}, \quad k = 1, 2, \dots, N;$$

similar to the previous case. At these points, obviously,

$$\cos p = \cosh \gamma(q), \quad \cosh \gamma(p) = \cos q, \quad \sinh^2 \gamma(p) = -\sin^2 q.$$

So, the integration contours in (4.8) may be squeezed and the coefficient $g_{2l+1}(r)$ is expressed through the sum of residua

$$g_{2l+1}(r) = \frac{e^{-(2l+1)/\Lambda}}{(2l+1)! N^{2l+1}} \sum_{[q]}^{(b)} \prod_{i=1}^{2l+1} \left(\frac{e^{-r\gamma_i - \eta_i}}{\sinh \gamma_i} \right) \frac{V_{2l+1}[q]}{\prod_{i < j}^{2l+1} \sinh^2((\gamma_i + \gamma_j)/2)}, \quad (4.11)$$

where

$$V_{2l+1}[q] = C_l^{2l+1} 2^{-2l(l+1)} \prod_{i=1}^{l-1} \prod_{j=i+1}^l (\cos q_{2i} - \cos q_{2j})^2 \times \\ \times \prod_{i=1}^l \prod_{j=i+1}^{l+1} (\cos q_{2i-1} - \cos q_{2j-1})^2 \prod_{i=1}^l \sin^2 q_{2i}. \quad (4.12)$$

The expression (4.12) differs from the corresponding ferromagnetic one (3.30). But it occurs that also in the paramagnetic case the symmetric part of the function $V_{2l+1}[q]$ – with respect to change of each variable with even subscript q_{2i} by each variable with odd subscript q_{2j-1}

(the number of the permutations is C_l^{2l+1}) – coincides with the even part of the function $U_{2l+1}[q]$

$$U_{2l+1}[q] = \prod_{i < j}^{2l+1} \sin^2\left(\frac{q_i - q_j}{2}\right).$$

The equality analogous to the eq. (3.31) takes place

$$V_{2l+1}^{\text{sim}}[q] = U_{2l+1}^{\text{even}}[q],$$

where

$$V_{2l+1}^{\text{sim}}[q] = \frac{1}{C_l^{2l+1}} \sum_{\mathbf{p}} V_{2l+1}[q],$$

$$U_{2l+1}^{\text{even}}[q] = 2^{-(2l+1)} \sum_{[\sigma=\pm 1]} \prod_{i < j}^{2l+1} \sin^2\left(\frac{\sigma_i q_i - \sigma_j q_j}{2}\right).$$

Consequently the function $V_{2l+1}[q]$ can be replaced by the function $U_{2l+1}[q]$ under the summation in the r.h.s. of eq. (4.11). We obtain eventually

$$n = 2l + 1,$$

$$g_n(r) = \frac{e^{-n/\Lambda}}{n! N^n} \sum_{[q]}^{(b)} \prod_{i=1}^n \left(\frac{e^{-r\gamma_i - \eta_i}}{\sinh \gamma_i} \right) F_n^2[q], \quad (4.13)$$

$$F_n[q] = \prod_{i < j}^n \frac{\sin((q_i - q_j)/2)}{\sinh((\gamma_i + \gamma_j)/2)}, \quad F_1[q] = 1. \quad (4.14)$$

The functions $\gamma_i = \gamma(q_i)$ and $\eta_i = \eta(q_i)$ are defined by the eq. (3.29).

The form of the coefficients $g_n(r)$ and of the function $F_n[q]$ is surprisingly the same both the ferro- and paramagnetic cases. In the ferromagnetic case the correlation function expansion is extended over even n and for the paramagnetic case – over odd n

$$\langle \sigma(\mathbf{r}_1) \sigma(\mathbf{r}_2) \rangle = (\xi \cdot \xi_T) e^{-r/\Lambda} \sum_{l=0}^{\infty} g_{2l}(r), \quad K > K_c; \quad (4.15)$$

$$\langle \sigma(\mathbf{r}_1) \sigma(\mathbf{r}_2) \rangle = \sinh\left(\frac{\gamma_0 + \gamma_\pi}{2}\right) (\xi \cdot \xi_T) e^{-r/\Lambda} \sum_{l=0}^{\infty} g_{2l+1}(r), \quad K < K_c. \quad (4.16)$$

Notice that the even values of the circumference N were assumed in the factorization procedure for the functions $P(z)$ and $Q(z^{-1})$ (see eqs. (3.4) and (4.3)). For the odd N the Brillouin zone does not contain the $q = \pi$ point in the boson case. On the contrary this value appears in the fermion spectrum. It is not difficult to account for this detail, it is irrelevant for the final result.

5 The scaling limit

The IM correlation length μ^{-1} diverges when the coupling parameter K tends to the critical value K_c . The scaling limit means that both the number of sites on the cylinder circumference and the distance between the correlating spins tend to infinity provided that the corresponding scaling variables are finite:

$$\begin{aligned} N &\rightarrow \infty, \quad r \rightarrow \infty, \quad \mu \rightarrow 0, \\ \mu N &= \nu = \text{const}, \quad \mu r = \rho = \text{const}. \end{aligned} \quad (5.1)$$

Contrary to the limit $N \rightarrow \infty$, $\mu = \text{const}$ the specific “cylinder quantities” $\ln \xi_T$, Λ^{-1} , $\eta(q)$ do not vanish in the limit (5.1). They survive and nontrivially depend on the scaling circumference ν .

Before the evaluation of these functions note that all summands in (4.13) contain the factors $\exp(-r\gamma(q))$ which restrict the summation area by small values of momenta $q \ll 1$ if $r \rightarrow \infty$. Consequently

$$\gamma(q) = 2 \text{arsinh} \sqrt{\frac{\mu^2}{4} + \sin^2 \frac{q}{2}} \simeq \sqrt{\mu^2 + q^2} \ll 1,$$

and we have in the limit (5.1), for instance,

$$\frac{1}{N} \sum_{q=-\pi}^{\pi} \frac{e^{-r\gamma(q)}}{\sinh \gamma(q)} = \sum_{q=-\infty}^{\infty} \frac{e^{-\rho\omega(q)}}{\nu\omega(q)} + o(\exp(-\rho/\mu^\epsilon)), \quad (5.2)$$

where ϵ is some positive constant $0 < \epsilon < 1$ and $\omega(q) = \sqrt{q^2 + 1}$. The sum in r.h.s. of (5.2) means

$$\sum_q^{(b)} v(q) = \sum_{k=-\infty}^{\infty} v(2\pi k/\nu).$$

The lattice form factor (4.14) in the limit $q \ll 1$ is reduced to the continuous one (1.3). It does not depend explicitly on circumference ν .

The calculation of eq. (3.13) gives the scaling limit for $\ln \xi_T$

$$\ln \xi_T = \frac{\nu^2}{2\pi^2} \int_1^\infty \frac{d\omega_1 d\omega_2}{\sinh(\nu\omega_1) \sinh(\nu\omega_2)} \ln \left| \frac{q_1 + q_2}{q_1 - q_2} \right|. \quad (5.3)$$

According to (3.14) and (3.15) one can obtain other representation for this quantity

$$\ln \xi_T = \nu^2 \int_1^\infty d\nu' \nu' c^2(\nu'), \quad c(\nu) = \frac{1}{\pi} \int_1^\infty \frac{d\omega}{q \sinh(\nu\omega)}. \quad (5.4)$$

The coherence length (3.7) in the scaling limit can be written in the following way

$$\Lambda^{-1} = \mu \Delta(\nu), \quad (5.5)$$

where

$$\Delta(\nu) = \frac{1}{\pi} \int_0^\infty dq \ln \coth(\nu\omega(q)/2) = \frac{\nu}{\pi} \int_1^\infty \frac{d\omega q}{\sinh(\nu\omega)}. \quad (5.6)$$

And for the last quantity (3.34) which is specific for the cylinder we obtain the following scaling limit expression

$$\eta(q, \nu) = \frac{2\omega(q)}{\pi} \int_0^\infty \frac{dp}{\omega^2(q) + p^2} \ln \coth(\nu\omega(p)/2). \quad (5.7)$$

The two dimensional Ising model can be considered as a lattice regularization of some quantum field model in (1+1) dimensional (euclidian) space-time. In our case one of the dimension is infinite and other is finite with periodical boundary condition along it. One can define the corresponding renormalized two-point equal time Green function at finite temperature and in infinite volume

$$G(\rho, \nu) = z^{-1} \langle \sigma(0) \sigma(r) \rangle.$$

Here z is the wave function renormalization constant. We shall argue below that in our case this constant is

$$z = 2\xi. \quad (5.8)$$

The connection between scaling variables ρ, ν and physical ones is tuned by the equations

$$\nu = m\beta, \quad \rho = mR,$$

where m is the renormalized field excitation mass, β is the inverse temperature and R is the spatial distance between correlating fields.

Collecting the corresponding formulas, we obtain the following form factor representation for the renormalized two-point equal time Green function at finite temperature

$$G(\rho, \nu) = \xi_T(\nu) e^{-\rho\Delta(\nu)} \sum_n g_n(\rho, \nu), \quad g_0 = 1, \quad (5.9)$$

$$g_n(\rho, \nu) = \frac{1}{n!} \sum_{[q]}^{(b)} \prod_{i=1}^n \left(e^{-\rho\omega_i - \eta_i} / \nu\omega_i \right) F_n^2[q], \quad (5.10)$$

$$F_n[q] = \prod_{i < j}^n \frac{q_i - q_j}{\omega_i + \omega_j}, \quad F_1 = 1, \quad (5.11)$$

$$\eta_i = \eta(q_i, \nu), \quad \omega_i = \omega(q_i) = \sqrt{q_i^2 + 1}, \quad \rho = mR, \quad \nu = m\beta;$$

$$n = 0, 2, 4, \dots \text{for the ferromagnetic case,}$$

$$n = 1, 3, 5, \dots \text{for the paramagnetic case.}$$

The functions $\xi_T(\nu)$, $\Delta(\nu)$ and $\eta(q, \nu)$ are defined by eqs. (5.3)–(5.7).

It is easy to see that all “cylinder corrections” (5.3)–(5.7) exponentially decrease when the circumference ν grows. The summation in (5.10) over phase volume turns into the integral if $\nu \rightarrow \infty$

$$\frac{1}{\nu} \sum_q^{(b)} \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} dq$$

and the expressions (5.9)–(5.11) turn into the classic form factor representation (1.1)–(1.3) for the case of infinite continuous plane (quantum field at zero temperature).

The form factor (5.11) is Lorentz invariant quantity. It can be rewritten explicitly in terms of Mandelstam variables

$$F_n[q] = \prod_{i < j}^n (1 - 4/s_{ij})^{1/2},$$

where

$$s_{ij} = (\omega_i + \omega_j)^2 - (q_i + q_j)^2.$$

Accounting for this property it is easy to calculate the Fourier transformation of the Green function

$$\tilde{G}(p^2) = \int d^2 R e^{ipR} G(\rho, \infty)$$

and to obtain the Lehmann representation for the propagator

$$\tilde{G}(p^2) = \frac{1}{2} \sum_n \tilde{g}_n(p^2), \quad (5.12)$$

$$\tilde{g}_n(p^2) = \int_{(nm)^2}^{\infty} \frac{dQ^2}{p^2 + Q^2} \Delta_n(Q^2), \quad Q^2 = Q_0^2 - Q_1^2, \quad (5.13)$$

$$\Delta_n(Q^2) = \frac{1}{n!(2\pi)^n} \int_{-\infty}^{\infty} \left(\prod_{i=1}^n \frac{dq_i}{\omega_i} \right) F_n^2[q] \delta\left(\sum_{i=1}^n q_i - Q_1\right) \delta\left(\sum_{i=1}^n \omega_i - Q_0\right). \quad (5.14)$$

For the paramagnetic case the summation in (5.12) is over odd n . In this case the function $\tilde{G}(p^2)$ possesses the transparent structure of singularities in the p^2 complex plane [21]. The first term in the expansion (5.12) $\tilde{g}_1(p^2)$ has the simple pole at $p^2 = -m^2$

$$\tilde{g}_1(p^2) = \frac{2}{p^2 + m^2}. \quad (5.15)$$

With the allowance of (5.15) the following normalization of the propagator

$$\lim_{p^2 \rightarrow -m^2} [\tilde{G}(p^2)(p^2 + m^2)] = 1$$

leads to the eq. (5.8) for the wave function renormalization constant. The next terms in (5.12) have the branch points at $p^2 = -(nm)^2$, corresponding to the thresholds of the n -particles intermediate states. The presence of the simple pole as the lowest singularity in

the propagator is the necessary and sufficient condition for the existence of the asymptotic states corresponding to observed particles.

The next important question is about the sort of the particles: more precisely, about the type of statistics (Bose or Fermi) of the particles in the asymptotic and intermediate states. To clarify the problem we have to compare the IM result with that for the free boson and/or free fermion fields. Let us recall the expressions for the pressure and energy density of the free boson and fermion relativistic gas in $(1+1)$ dimension

$$\frac{P^{(f,b)}}{m^2} = \frac{1}{\pi} \int_1^\infty \frac{d\omega q}{e^{\nu\omega} \pm 1}, \quad \frac{\mathcal{E}^{(f,b)}}{m^2} = \frac{1}{\pi} \int_0^\infty \frac{dq \omega}{e^{\nu\omega} \pm 1},$$

where $\nu = \beta\mu$, $\omega(q) = \sqrt{q^2 + 1}$; the signs $(+, -)$ correspond to fermion and boson respectively. The corresponding IM thermodynamic quantities are exactly the same as for the free fermion gas, i.e.

$$P^{(I)} = P^{(f)}, \quad \mathcal{E}^{(I)} = \mathcal{E}^{(f)}.$$

This was one of the reasons to interpret the IM as the free fermion system.

By the way, one can see that the specific ‘‘cylinder quantities’’ (5.4) and (5.6) may be expressed through the free Bose- and Fermi-gas thermodynamic characteristics

$$\begin{aligned} \Delta(\nu) &= \frac{\nu}{m^2} (P^{(b)} + P^{(f)}), \\ c(\nu) &= \frac{1}{m^2} (\mathcal{E}^{(b)} + \mathcal{E}^{(f)} - P^{(b)} - P^{(f)}). \end{aligned}$$

The relativistic gas is the system with the zero chemical potential: the number of particles is not fixed rather it is defined by the thermodynamic equilibrium condition. As the consequence, the difference between fermion and boson gas at low temperature is quantitatively negligible because the gas is dilute and becomes apparent only at high temperature. For example,

$$\frac{P^{(b)} - P^{(f)}}{P^{(b)} + P^{(f)}} \simeq \begin{cases} e^{-\nu}/2^{3/2} & \text{for } \nu \gg 1 \\ 1/3 & \text{for } \nu \ll 1. \end{cases} \quad (5.16)$$

The free field propagator $\tilde{G}^{(b,f)}(p^2)$ is the simple pole (is proportional to the simple pole for the fermions)

$$\tilde{G}^{(b,f)}(p^2) = \frac{1}{p^2 + m^2}, \quad p^2 = p_0^2 + p_1^2.$$

At finite temperature the zero component of the momentum becomes discrete

$$p_0^{(k)} = \begin{cases} 2\pi k/\beta & \text{for bosons} \\ 2\pi(k + 1/2)/\beta & \text{for fermions} \end{cases}$$

In result the spatial correlations of the free boson and fermion fields are

$$G^{(b,f)}(\rho, \nu) = \sum_q^{(b,f)} \frac{e^{-\rho\omega(q)}}{2\nu\omega(q)}. \quad (5.17)$$

The corresponding IM pole contribution is

$$G_{\text{pole}}^{(I)}(\rho, \nu) = \xi_T(\nu) e^{-\rho \Delta(\nu)} \sum_q^{(b)} \frac{e^{-\rho \omega(q) - \eta(q, \nu)}}{2\nu \omega(q)}. \quad (5.18)$$

At low temperature $\nu \gg 1$ the functions $\ln \xi_T(\nu)$, $\Delta(\nu)$ and $\eta(q, \nu)$ are exponentially small

$$\begin{aligned} c(\nu) &\simeq e^{-\nu} \sqrt{2/\pi\nu}, & \ln \xi_T(\nu) &\simeq \frac{\nu^2}{\pi} e^{-2\nu}, \\ \Delta(\nu) &\simeq e^{-\nu} \sqrt{2/\pi\nu}, & \eta(q, \nu) &\simeq \frac{4e^{-\nu}}{\omega(q) \sqrt{2\pi\nu}}, \end{aligned}$$

and the function $G_{\text{pole}}^{(I)}(\rho, \nu)$ coincides with that for the free boson field.

The high temperature behaviour is more instructive, as was noted above. This assertion, at first sight, contradicts to our nonrelativistic quantum physics experience: the lower temperature – the higher role of the quantum effects. In fact, the quantum corrections become apparent if the occupancy of the energy levels becomes large enough. If the particle density is fixed this is realized when the temperature tends to zero. But in the relativistic gas the particle density decreases exponentially, due to the annihilation, when the temperature diminishes. Hence, the occupation numbers tend to zero and the difference between the Bose and Fermi statistics is irrelevant. On the contrary, at high temperature the level occupancy depends on the competition between the extension of the accessible levels and the growing particle density. The quantitative issue of the described picture is given by the eq.(5.16) for the pressures. The same is true for the correlation functions. One can see from eq. (5.17) that for $\nu \gg 1$ (low temperature) the difference between boson and fermion correlation functions is quantitatively negligible. But for $\nu \ll 1$ (high temperature) the difference between boson and fermion field correlations is appreciable, as it is seen from the following equations

$$\begin{aligned} G^{(b)}(\rho, \nu) &\simeq \frac{1}{2\nu} e^{-\rho} + \frac{1}{2\pi} e^{-2\pi\rho/\nu}, \\ G^{(f)}(\rho, \nu) &\simeq \frac{1}{\pi} e^{-\pi\rho/\nu}. \end{aligned}$$

The high temperature asymptotic behaviour of the functions $\xi_T(\nu)$ is

$$\xi_T(\nu) \simeq \zeta \cdot \left(\frac{2}{\pi\nu} \right)^{1/4},$$

where

$$\zeta = \exp\left(-\frac{1}{4} \int_0^\infty \frac{dx}{x} e^{-2x} \tanh^2 x\right) \simeq 1.04. \quad (5.19)$$

Note that the integral in (5.19) can be expressed through the Glaisher's constant C_G so that

$$\zeta = \sqrt{\pi} e^{1/4} 2^{-1/6} C_G^{-3}.$$

The asymptotics of $\Delta(\nu)$ and $\eta(q, \nu)$ for $\nu \ll 1$ are

$$\Delta(\nu) \simeq \frac{\pi}{4\nu}, \quad e^{-\eta(q, \nu)} \simeq \frac{\nu\omega(q)}{2}.$$

In result we obtain from (5.18)

$$G_{\text{pole}}^{(I)}(\rho, \nu) \simeq \zeta\left(\frac{2}{\pi\nu}\right)^{1/4} e^{-\rho(1+\pi/4\nu)}. \quad (5.20)$$

One can see that the IM correlations (5.20) decrease slower than the free fermion gas correlations, if the temperature grows, but more fast (the screening) compare to the free boson gas correlations.

Therefore, on the level of the finite temperature Green function the quantum field system, corresponding to the IM scaling limit from the paramagnetic region, looks like strongly interacted bosons. The thermal fluctuations dress the intermediate states: the averaging over phase volume is accompanied by the additional thermal weight function $\exp(-\eta(q, \nu))$. But the dynamic quantities (form factors squared) are left bare. In spite of this the corresponding thermodynamic quantities (free energy, pressure, energy density) are identical to those of the free fermion gas at any temperature – the miracle in a sense.

The picture of the ferromagnetic region scaling limit is more sophisticated. First of all the nonvanishing vacuum expectation value at zero temperature appears $\langle\sigma\rangle \neq 0$. So, the Green function of the excitations over the condensate is to be defined. In our case it means that the first term in the expansions (5.9), (5.12) is to be removed and summation is expanded over $n = 2, 4, \dots$. The lowest singularity of the propagator is contained in $\tilde{g}_2(p^2)$ term of the Lehmann representation (5.12)–(5.14). It is not simple pole. Consequently there are no asymptotic states corresponding to the particles. From the other hand the thermodynamic quantities here are the same as in the paramagnetic case: the matter exists, the particles do not. This paradox is spurious. Due to the selfdual properties of the IM there exists side by side the order parameter $\langle\sigma\rangle$ the disorder parameter $\langle\mu\rangle$. The Green function corresponding to the disorder parameter in the ferromagnetic region is identical to that corresponding to the order parameter in the paramagnetic region. So, there exist the quantum field excitations corresponding to the particles states also in the ferromagnetic region. But these states cannot be generated by the external sources which are locally connected with the quantized field σ . Moreover, the corresponding sources apparently do not commute with the internal field. The picture seems to be instructive, in particular, for QCD confinement problem.

Note also that the correlation function on a cylinder in the ferromagnetic region $K > K_c$ vanishes if $\rho \rightarrow \infty$ at any finite circumference ν

$$\lim_{\rho \rightarrow \infty} G(\rho, \nu) = \begin{cases} 1/2 & \text{for } \nu = 0 \\ 0 & \text{for } \nu > 0 \end{cases}$$

Therefore, the IM field condensate appearing in the ferromagnetic region is not stable. Thermal fluctuations at any low temperature destroy them.

6 Conclusion

It should be observed first of all that, although the representations (4.15), (4.16) for the correlation function are obtained for different regions of the coupling parameter ($K > K_c$ and $K < K_c$ respectively), both of them are valid and equal to each other for all values of K at any finite cylinder circumference N . For the finite N the expansions (4.15), (4.16) are not series rather finite sums due to the form factor $F_n[q]$ is identically zero for $n > N$. Only the thermodynamic limit $N \rightarrow \infty$ splits the domain of their validity.

The obtained lattice form factor representations possess one beautiful feature: they can be checked numerically or analytically. For small values of N the result may be computed in other way, for example with the help of the transfer matrix technique. In particular at $N = 1$ we have the one dimensional Ising chain. The elementary calculation gives

$$\langle \sigma(0)\sigma(r) \rangle = t^r. \quad (6.1)$$

One can see after simple transformations that both the representations (4.15) and (4.16) coincide with (6.1).

Apart from its intrinsic interest the Ising model deserves scrutiny because of its relevance to nonperturbative explorations in quantum field theory. Mutual correspondence between field model and spin lattice system in scaling regime implies in our case the limit $K \rightarrow K_c$, $(N, M, r) \rightarrow \infty$ at $(N\mu, M\mu, r\mu) = \text{const.}$ Mysterious process of thermodynamic or scaling limit may be observed in detail proceeding from the explicit expressions obtained in this work.

The IM correlation function $\langle \sigma(\mathbf{r})\sigma(\mathbf{r}') \rangle$ in scaling limit corresponds to the quantum field model two point Green function $G(x, t; L, \beta)$. The variables of the Green function are assumed to be scaled on mass: $m = 1$, $x = \mu|r_x - r'_x|$ is the spatial distance, $t = \mu|r_y - r'_y|$ is the temporal one. $L = \mu M$ is the volume, $\beta = \mu N$ is the temperature. The boundary conditions along the temporal coordinate are quite definite: periodic for the boson field, antiperiodic for the fermion field. This is the issue of the appropriate formulation of the quantum field theory at finite temperature. On the contrary, the spatial boundary condition may be arbitrary. In any case the presence of boundaries breaks down explicitly the Lorentz invariance (the rotational invariance in Euclidian metric case). The spatial and temporal coordinates are not equivalent in general. In this sense the configuration of correlating spins considered in this paper corresponds to the equal time Green function $G(x, 0; \infty, \beta)$ in the infinite volume at finite temperature. With some stipulations this function may be viewed also as that at zero temperature in finite volume $\tilde{L} = \mu N$ (with periodicity in spatial coordinate). One has to keep in mind in addition that this is not equal time rather equal site Green function $G(0, \tilde{t}; \tilde{L}, \infty)$. It is necessary to consider the correlating spins placed on a cylinder circumference to compute the equal time Green function in finite volume and zero temperature. The corresponding problem may be solved by the method used in this work but with the Toeplitz determinant technique being properly modified.

Meanwhile, the representation (4.13)–(4.16) for the IM correlation function has such transparent structure that the conjecture for the case of general displacement of correlating

spins suggests itself. The simplest one is

$$\langle \sigma(0) \sigma(r_x, r_y) \rangle = (\xi \cdot \xi_T) e^{-r_x/\Lambda} \sum_n g_n(r_x, r_y), \quad (6.2)$$

$$g_n(r_x, r_y) = \frac{e^{-n/\Lambda}}{n! N^n} \sum_{[q]}^{(b)} \prod_{j=1}^n \left(\frac{e^{-r_x \gamma_j - i r_y q_j - \eta_j}}{\sinh \gamma_j} \right) F_n^2[q]. \quad (6.3)$$

It is amazing that this generalization is in exact agreement with the transfer matrix results for the N-rows Ising chains. We have checked it analytically for $N = 2, 3, 4$ and numerically for $N = 5, 6$.

I thank V.Shadura for the fruitful discussions and O.Lisovy for the assistance in computations.

This work was supported by the INTAS Program (Grant INTAS-97-1312).

Appendix

The elements of the Toeplitz matrix arranged along the diagonals are equal

$$A_{x,x'} = A_k \text{ for } k = x - x' = 0, \pm 1, \dots, \pm r - 1, \quad (A1)$$

$$A^{(r)} = \begin{pmatrix} A_0 & A_{-1} & A_{-2} & \cdots & A_{-r+1} \\ A_1 & A_0 & A_{-1} & \cdots & A_{-r+2} \\ A_2 & A_1 & A_0 & \cdots & A_{-r+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{r-1} & A_{r-2} & A_{r-3} & \cdots & A_0 \end{pmatrix}. \quad (A2)$$

The superscript in $A^{(r)}$ shows the matrix dimension. If there is the function of the complex variable $A(z)$ analytical in the ring $\alpha < |z| < \alpha^{-1}$ ($\alpha < 1$) such that the coefficients A_k of its Loran series

$$A(z) = \sum_{k=-\infty}^{\infty} A_k z^k \quad (A3)$$

coincide with the matrix element (A2) then the matrix (A1) can be represented in the form

$$A_{x,x'} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} z^{-x+x'} A(z). \quad (A4)$$

The function $A(z)$ in the integrand of (A4) is called the kernel of the Toeplitz matrix $A^{(r)}$.

If the inverse matrix is known then the calculation of the determinant reduces to calculation of trace with help of the identity

$$\ln \det A^{(r)}(\lambda) = \int d\lambda \operatorname{Sp} \left[(A^{(r)}(\lambda))^{-1} \frac{\partial}{\partial \lambda} A^{(r)}(\lambda) \right] + \text{const}, \quad (A5)$$

where λ is a parameter of which the matrix elements depend on explicitly.

The inversion procedure for the Toeplitz matrix simplifies if both the kernel (A3) and $\ln A(z)$ may be expanded in Loran series. If so, the kernel $A(z)$ can be expressed in factorized form

$$A(z) = P(z) \cdot Q(z^{-1}). \quad (\text{A6})$$

The functions $P(z)$ and $Q(z)$ are analytic inside the circle $|z| \leq \alpha^{-1}$:

$$\ln P(z) = \mathcal{P} \ln A(z), \quad \ln Q(z^{-1}) = \tilde{\mathcal{P}} \ln A(z). \quad (\text{A7})$$

The projection operators \mathcal{P} and $\tilde{\mathcal{P}}$ extract the analytic part of the given function $v(z)$ inside and outside the circle $|z| = 1$ correspondingly

$$\begin{aligned} \mathcal{P}^2 &= \mathcal{P}, \quad \tilde{\mathcal{P}}^2 = \tilde{\mathcal{P}}, \quad \mathcal{P}\tilde{\mathcal{P}} = \tilde{\mathcal{P}}\mathcal{P} = 0, \quad \mathcal{P} + \tilde{\mathcal{P}} = 1, \\ \mathcal{P}v(z) &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{d\xi v(\xi)}{\xi - z}, \quad \tilde{\mathcal{P}}v(z) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{d\xi v(\xi)}{z - \xi}. \end{aligned} \quad (\text{A8})$$

The problem of the Toeplitz matrix inversion can be reduced to solving the system of the Wiener-Hopf sum equations [16]. In terms of the projection operators (A8) the matrix elements of the inverse matrix $(A^{(r)})_{x,x'}^{-1}$ are

$$(A^{(r)})_{x,x'}^{-1} = \mathcal{P} \frac{z^{r-x}}{Q(z^{-1})} \tilde{\mathcal{P}} w^{-1}(z) [1 - \tilde{\mathcal{P}} w(z) \mathcal{P} w^{-1}(z)]^{-1} \mathcal{P} \frac{z^{x'}}{Q(z^{-1})} \Big|_{z=0}, \quad (\text{A9})$$

$$(A^{(r)})_{x,x'}^{-1} = \mathcal{P} \frac{z^{-x}}{P(z)} \mathcal{P} w(z) [1 - \mathcal{P} w^{-1}(z) \tilde{\mathcal{P}} w(z)]^{-1} \tilde{\mathcal{P}} \frac{z^{x'-r}}{P(z)} \Big|_{z=0}, \quad (\text{A10})$$

$$(A^{(r)})_{x,x'}^{-1} = \mathcal{P} \frac{z^{-x}}{P(z)} \left\{ 1 - \mathcal{P} [1 - w(z) \mathcal{P} w^{-1}(z) \tilde{\mathcal{P}}]^{-1} w(z) \mathcal{P} w^{-1}(z) \right\} \mathcal{P} \frac{z^{x'}}{Q(z^{-1})} \Big|_{z=0}, \quad (\text{A11})$$

where

$$w(z) = z^r W(z), \quad W(z) = \frac{P(z)}{Q(z^{-1})}. \quad (\text{A12})$$

Each of the alternative but equivalent representations (A9)–(A11) for the inverse matrix $(A^{(r)})_{x,x'}^{-1}$ are preferable for the calculation of different matrix elements. In particular, the eq. (A9) is suitable for the calculation of the element $(A^{(r)})_{r-1,0}^{-1}$, the eq. (A11) – for the element $(A^{(r)})_{0,0}^{-1}$:

$$(A^{(r)})_{r-1,0}^{-1} = \frac{1}{Q^2(0)} \left\{ z \tilde{\mathcal{P}} w^{-1}(z) [1 - \tilde{\mathcal{P}} w(z) \mathcal{P} w^{-1}(z)]^{-1} \right\} \Big|_{z \rightarrow \infty}, \quad (\text{A13})$$

$$(A^{(r)})_{0,0}^{-1} = \frac{1}{P(0)Q(0)} \left\{ 1 - \mathcal{P} [1 - w(z) \mathcal{P} w^{-1}(z) \tilde{\mathcal{P}}]^{-1} w(z) \mathcal{P} w^{-1}(z) \right\} \Big|_{z=0}. \quad (\text{A14})$$

Expanding the inverse operators that are contained in the r.h.s. of (A13) and (A14) into geometrical progression

$$[1 - \tilde{\mathcal{P}} w(z) \mathcal{P} w^{-1}(z)]^{-1} = \sum_{l=0}^{\infty} [\tilde{\mathcal{P}} w(z) \mathcal{P} w^{-1}(z)]^l,$$

$$[1 - w(z)\mathcal{P}w^{-1}(z)\tilde{\mathcal{P}}]^{-1} = \sum_{l=0}^{\infty} [w(z)\mathcal{P}w^{-1}(z)\tilde{\mathcal{P}}]^l$$

and using the integral representations for the projection operators \mathcal{P} , $\tilde{\mathcal{P}}$ (A8) one can obtain for (A13) with allowance (A12)

$$(A^{(r)})_{r-1,0}^{-1} = \frac{1}{Q^2(0)} \sum_{l=0}^{\infty} a_{2l+1}(r), \quad (\text{A15})$$

$$a_{2l+1}(r) = \frac{1}{(2\pi i)^{2l+1}} \oint_{|z_i|<1} \frac{\prod_{i=1}^{2l+1} (dz_i z_i^r)}{z_1 z_{2l+1} \prod_{i=1}^{2l} (1 - z_i z_{i+1})} \frac{\prod_{i=1}^l W(z_{2i})}{\prod_{i=0}^l W(z_{2i+1}^{-1})}. \quad (\text{A16})$$

In particular, the coefficients $a_1(r)$ and $a_3(r)$ are

$$a_1(r) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz z^{r-2}}{W(z^{-1})},$$

$$a_3(r) = \frac{1}{(2\pi i)^3} \oint_{|z_i|<1} \frac{d(z_1 z_2 z_3) (z_1 z_2 z_3)^r}{z_1 z_3 (1 - z_1 z_2) (1 - z_2 z_3)} \frac{W(z_2)}{W(z_1^{-1}) W(z_3^{-1})}.$$

Analogously one can obtain for the eq. (A14)

$$(A^{(r)})_{0,0}^{-1} = \frac{1}{P(0)Q(0)} \left(1 - \sum_{l=1}^{\infty} a_{2l}(r) \right), \quad (\text{A17})$$

$$a_{2l}(r) = \frac{1}{(2\pi i)^{2l}} \oint_{|z_i|<1} \frac{\prod_{i=1}^{2l} (dz_i z_i^r)}{z_1 z_{2l} \prod_{i=1}^{2l-1} (1 - z_i z_{i+1})} \frac{\prod_{i=1}^l W(z_{2i-1})}{\prod_{i=1}^l W(z_{2i}^{-1})} \quad (\text{A18})$$

and, in particular,

$$a_2(r) = \frac{1}{(2\pi i)^2} \oint_{|z_i|<1} \frac{d(z_1 z_2) (z_1 z_2)^r}{z_1 z_2 (1 - z_1 z_2)} \frac{W(z_1)}{W(z_2^{-1})},$$

$$a_4(r) = \frac{1}{(2\pi i)^4} \oint_{|z_i|<1} \frac{d(z_1 z_2 z_3 z_4) (z_1 z_2 z_3 z_4)^r}{z_1 z_4 (1 - z_1 z_2) (1 - z_2 z_3) (1 - z_3 z_4)} \frac{W(z_1) W(z_3)}{W(z_2^{-1}) W(z_4^{-1})}.$$

Consider now the auxiliary Toeplitz matrix $A^{(r)}(\lambda)$ depending on the parameter λ ,

$$A_{x,x'}(\lambda) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} z^{-x+x'} P(z) Q^\lambda(z^{-1}), \quad 0 \leq \lambda \leq 1, \quad (\text{A19})$$

where the functions $P(z)$ and $Q(z)$ are the same as in the kernel of the matrix $A^{(r)}$. It is seen from the definition (A19) that at $\lambda = 1$

$$A_{x,x'}(1) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} z^{-x+x'} P(z) Q(z^{-1}) = A_{x,x'},$$

and at $\lambda = 0$

$$A_{x,x'}(0) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} z^{-x+x'} P(z). \quad (\text{A20})$$

The matrix (A20) is the triangular one: all its elements to the right of the main diagonal are equal to zero

$$A_{x,x}(0) = P(0), \quad A_{x,x'}(0) = 0 \quad \text{if } x' > x.$$

Therefore

$$\det A^{(r)}(0) = P^r(0). \quad (\text{A21})$$

The integration constant in (A5) is defined with help of (A21).

Let us select the contribution $h(r)$ in $\ln \det A^{(r)}$ which does not vanish when the matrix dimension grows

$$\ln \det A^{(r)} = h(r) + \Delta h(r), \quad \Delta h(r) \rightarrow 0 \quad \text{if } r \rightarrow \infty.$$

It is not difficult to observe that only the first term of the inverse operator expansion in (A10) contributes in $h(r)$. This fact results in drastic simplification of corresponding computations. The inverse matrix (A10) is

$$\begin{aligned} (A^{(r)}(\lambda))_{x,x'}^{-1} &= \mathcal{P} \frac{z^{-x}}{P(z)} \mathcal{P} \frac{z^r P(z)}{Q^\lambda(z^{-1})} \tilde{\mathcal{P}} \frac{z^{x'}}{P(z)} = \\ &= \frac{1}{(2\pi i)^3} \oint_{|z_2| > |z_{1,3}|} \frac{d(z_1 z_2 z_3) z_1^{-x} (z_2/z_3) z_3^{-x'}}{z_1(z_2 - z_1)(z_2 - z_3)} \frac{P(z_2)}{P(z_1) Q^\lambda(z_2^{-1}) P(z_3)}. \end{aligned}$$

The derivative of the matrix (A19) is

$$\frac{\partial}{\partial \lambda} A_{x,x'}^{(r)}(\lambda) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} z^{-x+x'} P(z) Q^\lambda(z^{-1}) \ln Q(z^{-1}),$$

and finally one has for the trace in (A5)

$$\begin{aligned} \text{Sp} \left[(A^{(r)}(\lambda))^{-1} \frac{\partial}{\partial \lambda} A^{(r)}(\lambda) \right] &= \frac{1}{(2\pi i)^4} \oint_{|z_{2i}| > |z_{2j-1}|} \prod_{i=1}^4 dz_i \times \\ &\times \frac{[1 - (z_4/z_1)^r][1 - (z_3/z_4)^r](z_2/z_3)^r}{(z_2 - z_1)(z_2 - z_3)(z_4 - z_1)(z_4 - z_3)} \cdot \frac{P(z_2)P(z_4)Q^\lambda(z_4^{-1}) \ln Q(z_4^{-1})}{P(z_1)Q^\lambda(z_2^{-1})P(z_3)} = \\ &= r \ln Q(0) + \frac{1}{2\pi i} \oint_{|z|=1} dz \ln Q(z^{-1}) \frac{\partial}{\partial z} \ln P(z) + o(r^{-1}). \end{aligned} \quad (\text{A22})$$

It is seen from (A22) that nonvanishing contribution in the trace occurs to be independent on the parameter λ . Therefore, with allowance (A21) we obtain

$$h(r) = r[\ln P(0) + \ln Q(0)] + \frac{1}{2\pi i} \oint_{|z|=1} dz \ln Q(z^{-1}) \frac{\partial}{\partial z} \ln P(z). \quad (\text{A23})$$

For the calculation of $\Delta h(r)$ the eq. (A5) may be used by keeping the next terms in the expansion of (A10). But it is more convenient for this purpose to exploit the specific property of the Toeplitz matrix i.e.

$$(A^{(r+1)})_{0,0}^{-1} = \frac{\det A^{(r)}}{\det A^{(r+1)}}.$$

Therefore

$$\det A^{(r)} = \det A^{(r+k)} \prod_{s=r+1}^{r+k} (A^{(s)})_{0,0}^{-1},$$

and after substitution (A17) we obtain

$$\det A^{(r)} = e^{h(r+k)+\Delta h(r+k)} [P(0)Q(0)]^{-k} \prod_{s=r+1}^{r+k} \left(1 - \sum_{l=1}^{\infty} a_{2l}(s)\right). \quad (\text{A24})$$

It follows from (A23) that

$$e^{h(r+k)} = [P(0)Q(0)]^k e^{h(r)},$$

so, the factors $P(0)$ and $Q(0)$ in (A24) cancel. Due to this fact the limit $k \rightarrow \infty$ in (A24) can be taken

$$\det A^{(r)} = G(r) \cdot e^{h(r)}, \quad (\text{A25})$$

where

$$G(r) = \prod_{s=r+1}^{\infty} \left(1 - \sum_{l=1}^{\infty} a_{2l}(s)\right). \quad (\text{A26})$$

Expanding the product of sums in the r.h.s. of (A26)

$$G(r) = 1 - \sum_{s_1=r+1}^{\infty} \left(\sum_{l=1}^{\infty} a_{2l}(s_1)\right) + \sum_{s_1=r+1}^{\infty} \sum_{s_2=s_1+1}^{\infty} \left(\sum_{l=1}^{\infty} a_{2l}(s_1)\right) \left(\sum_{l=1}^{\infty} a_{2l}(s_2)\right) - \dots, \quad (\text{A27})$$

summing up over s_i and collecting the terms with the definite multiplicity of integration, one can represent (A27) in the form

$$G(r) = \sum_{l=0}^{\infty} g_{2l}(r). \quad (\text{A28})$$

The coefficients $g_{2l}(r)$ can be obtained from (A27) directly but it is more simple to compute them by use the following recursion relation

$$g_{2l}(r) + \sum_{k=0}^{l-1} \sum_{s=r+1}^{\infty} g_{2k}(s) a_{2(l-k)}(s) = 0. \quad (\text{A29})$$

It is not difficult to deduce this relation from (A27). Using the eq. (A18) for $a_{2l}(s)$ and initial condition $g_0(s) = 1$ one can obtain from (A29)

$$g_{2l}(r) = \frac{(-1)^l}{l!l!(2\pi i)^{2l}} \oint_{|z_i|<1} \frac{\prod_{i=1}^{2l} (dz_i z_i^r) \prod_{i=1}^{l-1} \prod_{j=i+1}^l [(z_{2i-1} - z_{2j-1})^2 (z_{2i} - z_{2j})^2]}{\prod_{i=1}^l \prod_{j=1}^l (1 - z_{2i-1} z_{2j})^2} \frac{\prod_{i=1}^l W(z_{2i-1})}{\prod_{i=1}^l W(z_{2i}^{-1})}, \quad (\text{A30})$$

in particular,

$$g_2(r) = -\frac{1}{(2\pi i)^2} \oint_{|z_i|<1} \frac{d(z_1 z_2) (z_1 z_2)^r}{(1 - z_1 z_2)^2} \frac{W(z_1)}{W(z_2^{-1})},$$

$$g_4(r) = -\frac{1}{4(2\pi i)^4} \oint_{|z_i|<1} \frac{d(z_1 z_2 z_3 z_4) (z_1 z_2 z_3 z_4)^r (z_1 - z_3)^2 (z_2 - z_4)^2}{(1 - z_1 z_2)^2 (1 - z_2 z_3)^2 (1 - z_3 z_4)^2 (1 - z_4 z_1)^2} \frac{W(z_1) W(z_3)}{W(z_2^{-1}) W(z_4^{-1})}.$$

Note that the integrand of (A30) is symmetric both with respect to permutations of the even subscript variables and with respect to permutations odd subscript variables.

The structure of Toeplitz matrix is altered in paramagnetic region. Here it is necessary to evaluate the determinant of the matrix which has the following form

$$B^{(r)} = \begin{pmatrix} A_{-1} & A_0 & A_1 & \cdots & A_{r-2} \\ A_{-2} & A_{-1} & A_0 & \cdots & A_{r-3} \\ A_{-3} & A_{-2} & A_{-1} & \cdots & A_{r-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{-r} & A_{-r+1} & A_{-r+2} & \cdots & A_{-1} \end{pmatrix}, \quad (\text{A31})$$

$$B_{x,x'} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} z^{-x+x'-1} A(z).$$

The technique stated above is not valid because the kernel of the matrix $B^{(r)}$

$$B(z) = z^{-1} A(z)$$

does not satisfy the Loran expansion condition for $\ln B(z)$ if $\ln A(z)$ does satisfy. Nevertheless, the problem can be reduced to that considered above. Really, it follows from the definition of the inverse matrix element through the determinant and cofactor

$$\det B^{(r)} = (-1)^r (A^{(r+1)})_{r,0}^{-1} \det A^{(r+1)}.$$

Using the eqs. (A15), (A16) for the inverse matrix and the eqs. (A25), (A28), (A30) for the determinant one can obtain

$$\det B^{(r)} = (-1)^r e^{h(r+1)} F(r), \quad (\text{A32})$$

where

$$F(r) = G(r+1) \cdot (A^{(r+1)})_{r,0}^{-1} = \frac{1}{Q^2(0)} \sum_{l=0}^{\infty} f_{2l+1}(r), \quad (\text{A33})$$

$$f_{2l+1}(r) = \sum_{k=0}^l a_{2k+1}(r+1) g_{2(l-k)}(r+1). \quad (\text{A34})$$

All summands in the r.h.s. of (A34) are the integrals of $2l+1$ multiplicity. After reduction the integrands to common denominator and symmetrization over even subscript variables and over odd subscript variables one can obtain

$$f_{2l+1}(r) = \frac{(-1)^l}{l!(l+1)!(2\pi i)^{2l+1}} \oint_{|z_i|<1} \prod_{i=1}^{2l+1} (dz_i z_i^r) \times \\ \times \frac{\prod_{i=1}^{l-1} \prod_{j=i+1}^l (z_{2i} - z_{2j})^2 \prod_{i=1}^l \prod_{j=i+1}^{l+1} (z_{2i-1} - z_{2j-1})^2 \prod_{i=1}^l (z_{2i} W(z_{2i}))}{\prod_{i=1}^l \prod_{j=0}^l (1 - z_{2i} z_{2j+1})^2 \prod_{i=0}^l (z_{2i+1} W(z_{2i+1}^{-1}))}, \quad (\text{A35})$$

and, in particular,

$$f_1(r) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz z^r}{z W(z^{-1})}, \\ f_3(r) = -\frac{1}{2(2\pi i)^3} \oint_{|z_i|<1} \frac{d(z_1 z_2 z_3) (z_1 z_2 z_3)^r (z_1 - z_3)^2}{(1 - z_1 z_2)^2 (1 - z_2 z_3)^2 (1 - z_3 z_1)^2} \frac{z_2 W(z_2)}{z_1 z_3 W(z_1^{-1}) W(z_3^{-1})}.$$

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